Physico–Mathematical Interactions: The Chern–Simons Story

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The essential role played by Chern–Simons terms in a variety of physical models provides yet another illustration of the unexpected but profound interactions between the two disciplines.

Ludwig Faddeev is one of the rare few who are regarded as mathematicians by mathematicians and as physicists by physicists (more commonly, it is vice versa!). He has contributed to the synthesis between our two disciplines in many domains; I had the pleasure of working with him on problems in general relativity and of many discussion over the years.

I propose to illustrate this synthesis through a particular set of examples, Chern–Simons (CS) "effects" in physics. This should both reflect the interplay of the two disciplines, as well as the uncanny way mathematical constructs become incorporated into physics (and sometimes even require the physicist to be a little precise). I must of necessity be succinct here, and refer (also compactly) to the literature for details. I shall not have the space here to illustrate the "backreaction", how such borrowing by physics in turn stimulates new mathematics; "CS mathematics" distinctly picked up after the advent of CS physics.

To do justice to the full web of interconnections involving (CS) terms [1] in physics would require one of those complicated tree (or loop) diagrams. I will have to omit entirely any discussion of some of the principal ones, for example the relation of CS to a) conformal field theory [2] (descending from 3 to 2 dimension in particular) b) anomalies [3], via its Pontryagin $F \wedge F$ ancestor (ascending from 3 to 4), c) to integrable systems [4], and d) in the currently hot AdS "M-theory" context. Instead, I will stick to some more concrete applications in which I have been involved.

The first sighting of CS in physics may have been in 1978, when D=11 supergravity (now back in a central role after two decades, thanks to M theory) was constructed. It arose there as a strange but unavoidable term needed for consistency of the theory, by preserving its local supersymmetry, then rapidly invaded lower dimensional, 4<D<11, models [5]. That a metric-independent, "topological", term (as physicists sometimes call them) should come to the rescue of a gravitational model is the first example of its uncanny properties! The theory necessarily contains a 3-form potential A, and it was found that there has to be an addition ${}_{11}I_{CS}[A] = \kappa \int A \wedge F \wedge F$, $F \equiv dA$ to the usual F^2 kinetic term in the action. The Einstein gravitational constant κ appears here, but not (of course) the metric. A smaller paradox is that despite appearances, I_{CS} is both parity and T even. From a physical point of view, this term generates a cubic self-interaction of the form field that is in fact essential in constructing its supersymmetry-preserving invariants [6]. These invariants are important as they can serve both as a check of M-theory currently thought to incorporate the D=11 theory as a limiting case and as counter terms in higher loop corrections to this maximal supergravity itself [6, 7]. This first physics appearance of CS passed relatively unnoticed for several reasons, not least the cubic nature of ${}_{11}I_{CS}[A]$, so that it did not directly affect the kinematics. Soon afterwards, and with no apparent connection to the above, the possibility and interest of introducing the 1-form CS term in spacetime dimensions D=3 was suggested by several authors [8, 9]. This time the context was more auspicious both because D=3 is closer to D=4 and

because physics in this planar world may even have observable consequences, in condensed matter settings as well as in high temperature limits of our D=4 world. Most of all, the interest was due to the fact that CS is here quadratic (and P, T violating), ${}_3I_{CS}[A] = \mu \int A \wedge F$ and hence can affect free-field (Maxwell) electromagnetism, and indeed lead to a finite-range but still gauge-invariant model. In its nonabelian incarnation, where A is a Lie algebra-valued 1-form, the term has the remarkable property that its numerical coefficient must be quantized for the quantum theory to be well-defined [9]. This idea, coming entirely from homotopy analysis, was of course a revelation to physicists on how *a priori* arbitrary parameters could in fact be restricted in their possible values (and hence had better also be renormalized by integer amounts only).

Before we consider some of the novel consequences of CS in this D=3 context, we first mention a quite different direction that gave rise to an enormous literature on so-called topological quantum field theories, including D=3 gravity, as we shall see. For the moment, we take the geometry to be Minkowskian $R^1 \times R^2$, to avoid global and topological complications. Then the Euler–Lagrange equations of purely CS actions simply become ${}^*F^\mu = 0$, in the absence of sources or ${}^*F^\mu = j^\mu$ when charged currents are present (${}^*F$ is the dual field strength, a 1–form). Thus the field is locally pure gauge wherever there are no sources; to find the general global solution with the properties that the field strength is equal to the current and vanishes elsewhere is then an interesting exercise. This is even more so in the nonabelian case where the abelian part is supplemented by the famous $\frac{1}{3} tr \int (A \wedge A \wedge A)$ addition to yield the same (but now nonabelian) Euler–Lagrange equation ${}^*F = 0$. Now in D=3, general relativity has a very similar property: spacetime is flat in the absence of source, since Einstein (G) and Riemann (R) tensors are equivalent, obeying the double-duality identity $G \equiv {}^*R^*$. Hence the Einstein equations $G = 0$ imply local flatness. Classification of such Minkowski signature locally flat, (or more generally locally constant curvature if there is also a cosmological constant so that $G + \Lambda g = 0$), spacetimes [10] has also become a physical industry of its own (we cannot even start to cite this literature). Here the physics involves global matching of flat patches at particle trajectories where the sources $T_{\mu\nu}$ (and therefore curvature) do not vanish. This "zero field-strength" field equation in source-free gravitational regions is of course very reminiscent of the above Yang–Mills CS story and indeed there is a CS form of (except for some fine print) Einstein gravity [11]. This insight has led to another large topic of its own ever since, namely the uses of the "antigeometrical" CS as geometry! [For a review of these formulations and their properties as well as references, see [12]]. There is also a direct mathematical connection, namely that between the Riemann–Hilbert problem and D=3 gravity coupled to several moving particles [13].

This is by no means the end of the gravity-CS interaction. There also exists in D=3 a genuine *gravitational* CS term

$$I_{CS} \equiv -\frac{1}{4} \int d^3x \, tr \epsilon^{\mu\nu\alpha} [R_{\mu\nu}\omega_\alpha + \frac{2}{3}\omega_\mu\omega_\nu\omega_\alpha] \,, \tag{1}$$

where the traces over the local Lorentz indices and ω_μ is the spin connection. Its variation yields the Cotton tensor $\sqrt{g}\, C^{\mu\nu} \equiv \epsilon^{\mu\alpha\beta} D_\alpha (R^\nu_\beta - \frac{1}{4}\delta^\nu_\beta R)$ (invented incidentally by a French mathematician whose brother was a physicist). This tensor is (despite appearances) symmetric, conserved and traceless; it is in fact the conformal curvature tensor in D=3, where the usual Weyl tensor vanishes. A pure CS action, whose sources must necessarily be traceless (and so cannot be particles or photons) is therefore not so interesting; it has exterior solutions to the third order equation $C_{\mu\nu} = 0$ that are conformally flat. Far more interesting is topologically massive gravity (TMG) [9], the sum of Einstein and gravitational CS actions. This model is in a sense the opposite of the topologically massive electrodynamics to be discussed below: the highest (third) derivative is in the CS part here. Despite the higher derivative content, TMG is perfectly ghost-free and consists of a massive

helicity ± 2 excitation according to the relative sign of the coefficients between the two terms; here P and T are each violated. TMG differs from the non-abelian TM vector case in *not* requiring quantization of the CS coefficient, despite seemingly similar arguments; this is quite paradoxical at first sight. Indeed, TMG still presents some other challenges; in particular no one has yet found the "Schwarzschild solution" for this nonlinear theory, although there is an amusing anyonic structure in the linear theory's solutions [14], with the CS term "twisting" the spin of source particle, as for the vector case. Other challenges include deciding whether the theory is renormalizable or not [15]; it might appear to be so on the basis of its higher derivative structure, but the latter does not quite "shield" the conformal contributions. The issue can in principle be decided using homotopy arguments. The interest here is that this is the only unitary but higher derivative theory of gravitation that has a dimensional coupling constant and would be the only finite quantum gravity model. There are of course (higher order in curvature) generalizations of the Cotton tensor to other odd dimensions, but they do not affect the propagators; there are generalizations to higher spin fields in D=3 as well (for s=3, see [15]).

Returning to physical applications of the plain abelian CS term, let me sketch a few of the reasons for their interest. First, if we add the usual Maxwell action to CS, the resulting topologically massive electrodynamics (TME) represents a single local degree of freedom, paradoxically endowed with a finite range but still gauge invariant. [This seems a very different way to get a finite mass than the Higgs mechanism but things are even more interesting; see [17]]. As background, recall that a pure Maxwell excitation in any dimension has (D-2) local excitations, the transverse spatial polarizations, while the gauge-broken (Proca theory) massive version has (D-1) of them. Further, the latter theories represent excitations of unit intrinsic angular momentum or spin. In D=3, however, it turns out that massless fields, including Maxwell, are (unlike massive ones) entirely devoid of spin [18] (but neutrinos still can have fermi statistics). The TME action inherits from pure Maxwell one local excitation; the CS input is to provide mass and thereby spin to this excitation. Here the CS term does break parity: the two degrees of the normal Proca theory are equivalent to a pair of "mirror" TME models. The TME field equations $dF + \mu {}^*F = 0$ are readily seen to imply that the field strength obeys Klein–Gordon propagation equations with μ representing the finite range. This mixing of normal metric and topological terms is what makes these models so different from the usual even-dimensional ones.

Mathematically, we have mentioned the role played by homotopy in CS physics. In fact there are several different roles, as we shall see. One is the cited quantization of the CS coefficient in the nonabelian theory: because the exponential of the action, $e^{iI/\hbar}$ is the basic quantum mechanical object there, actions must be invariant mod 2π under gauge transformations. Tracing the Π_3/Π_1 properties of CS under large gauge transformations shows it changes by a winding number so that its coefficient is necessarily integer; this is the dimensionless combination μ/g^2 where g^2 is the (dimensional in D=3) self-coupling constant. Another effect is that of the topology of planar configuration space – this is related to "anyons" or the loss of the standard spin-statistics relation in planar field theories [19]; it too can be represented in CS language [20].

My final example is the most recent; it deals with the definition and role of abelian CS in nontrivial topologies (the nonabelian CS story is still in progress [21]). The important issues already appear in cases as simple as $S^1 \times \Sigma^2$, finite-temperature ($\beta = \frac{1}{kT}$ is the perimeter of S^1) planar electrodynamics with (necessarily quantized) magnetic flux in the closed 2-manifold Σ^2. For details and earlier literature, I must refer to [22]. It is known that the naive CS term $\int A \wedge F$ now requires corrections to remain well-defined. These corrections to CS, and its behavior under large (not reducible to the identity) gauge transformations can in fact be elucidated in two complementary

ways, (and different from the known cohomology procedures cited in [23]). The first uses a classic result, the Chern–Weil theorem, which in D=4 tells us (using transgression) that for two different connections $(A, \tilde{A})$ on a bundle, $F \wedge F - \hat{F} \wedge \hat{F} = d[(A - \hat{A}) \wedge (F + \hat{F})]$. This provides a correct definition of I_{CS} on non-trivial bundles and also tells us that, unlike the simple-minded CS, it changes under large gauge rotations as the product of their "winding number" and the magnetic flux so as to respect the quantum action requirements mentioned above. The second way to reach the correct definition is – surprisingly – to embed the abelian model in a nonabelian SU(2) where all D=3 bundles are trivial; the fact that the homotopies of U(1) and SU(2) are opposite (Π_1 of the former and Π_3 of the latter fail to vanish) is no obstacle. [There is a third, heuristic, way – the one a desperate physicist would use to "guarantee" correctness when all else fails [23, 22], but I do not discuss it here!] I cite this seemingly pedantic formal discussion of CS definition precisely because what is the correct one in topologically nontrivial backgrounds has led to an immense and rather confused physics literature; confused because based on the naive $\int A \wedge F$, immense because it concerns the physically important "thermal" quantum electrodynamics where time is replaced by temperature through periodic identification of t, as we have mentioned. Now the CS miracle here is that, whether or not there is a "primitive" CS term in the original action (or indeed any action at all for the electromagnetic field A), there will arise an effective theory of A if one integrates out the charge particles that (necessarily) couple to it. In particular, if we have massive charged electrons obeying the usual Dirac equation $(\not{D} + m)\psi = 0$, $\not{D} \equiv \gamma(\partial + iA)$, then the (logarithm of the) determinant of the Dirac operator is essentially the functional that defines the effective action $I_{eff}[A]$. Since a fermion mass term is parity (and T) violating in D=3, there should naturally be CS terms in I_{eff}. Now the route to this action involves a careful process of first defining the determinant, *e.g.*, by ζ-function regularization. This enables one to expand in Seeley–deWitt coefficients, and find the correct, automatically gauge-invariant $I_{eff}[A]$. In particular the CS term always enters in a way that preserves invariance namely as part of deeper η-function structures. It was neglecting or omitting this necessary complication that gave rise to paradoxes involving large gauge transformations, that of course do *not* leave CS invariant. Indeed, to a physicist CS is basically the reminder already in the abelian but globally non-trivial space context that there is a further, discrete, gauge invariant variable besides the field strengths, namely the so-called flat connection rather than CS itself.

I have tried to give one short glimpse of how a theoretical physicist is often forced to use – and greatly benefits from – *a priori* far-removed mathematical constructs, a process that ultimately leads to further advances in mathematics as well.

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